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1982 J. Phys. A: Math. Gen. 15 587

(http://iopscience.iop.org/0305-4470/15/2/025)

J. Phys. A: Math. Gen. **15** (1982) 587–598. Printed in Great Britain

Pure and dilute $Z(N)$ spin and generalised gauge lattice systems: duality and criticality

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Received 21 May 1981, in final form 5 August 1981

Abstract. We consider the pure $Z(N)$ spin systems (including the standard Ising and Potts models) as well as generalised gauge systems (plaquettes or more complex simplex) in d-dimensional hypercubic lattices. These models are self-dual, and we show how this duality can be thought of as a series-parallel transformation. The simplicity of the equations enables conjectures to be made on the approximate critical frontier of the diluted version of the above systems, including some particular asymptotic behaviours which we believe to be exact. As an illustration the $d=2$ diluted $Z(4)$ spin system is discussed in some detail: for those regions where exact results are available the agreement is satisfactory.

1. Introduction

Since the Kramers and Wannier (1941) discussion of the Ising model, duality arguments have been a powerful tool for discussing the location of critical frontiers (CF) in various statistical systems such as bond percolation (Sykes and Essam 1963) and the N-states Potts model (Potts 1952, Kim and Joseph 1974). Quite general results (concerning in particular the $Z(N)$ models) have been obtained by Wu and Wang (1976), Alcaraz and Köberle (1980, 1981) and Savit (1980).

On different grounds Domb (1960), Nelson and Fisher (1975) and Yeomans and Stinchcombe (1979) (among others) have used in the discussion of Ising models a convenient variable, namely $t \equiv \tanh J/k_B T$ (J is the exchange coupling constant and T the temperature), referred to hereafter as *transmissivity* (Tsallis and Levy 1980, Levy *et al* 1980). This quantity can be extended (Tsallis and Levy 1981, Tsallis 1981) to cover the N-states Potts model; its expression is given by

$$t \equiv \frac{1-\exp(-NJ/k_B T)}{1+(N-1)\exp(-NJ/k_B T)}. \tag{1}$$

Remark that in the limit $N \to 1$, t equals $1-\exp(-J/k_B T)$, thus reproducing the variable which establishes the isomorphism with the bond percolation problem (Kasteleyn and Fortuin 1969).

The main advantage of the t variable is to provide a probability-like algorithm to calculate the equivalent transmissivity t_s of a *series* array of two bonds whose transmissivities are t_1 and t_2, namely

$$t_s = t_1 t_2. \tag{2}$$

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If the array is a *parallel* one the equivalent transmissivity t_p satisfies

$$t_p^D = t_1^D t_2^D \tag{3}$$

where

$$t_i^D \equiv \frac{1-t_i}{1+(N-1)t_i} \qquad (i=1,2,p). \tag{4}$$

The superscript D stands for *dual* (we refer here to the standard duality: see § 2); let us stress that through transformation (4), the series and parallel composition algorithms (respectively equations (2) and (3)) become *one and the same.*

In § 2 we extend the transmissivity to cover spin (see Wu and Wang 1976) and generalised gauge $Z(N)$ systems which contain several coupling constants, and we exhibit that the standard dual transformation can be very simply expressed as a series-parallel transformation.

The simplicity of equations (2) and (3) has enabled quite satisfactory suggestions (Tsallis and Levy 1980, Levy *et al* 1980, Tsallis 1981) to be made about the CF of the bond-dilute (or even bond-mixed) Ising and Potts models. It seems therefore quite natural to put forward (§ 3) analogous suggestions for the CF of diluted versions of general $Z(N)$ systems (only d-dimensional hypercubic lattices are considered). The particular case of the $d=2$, $Z(4)$ spin system is treated in detail: some already known numerical results are exactly or approximately recovered and a few predictions are proposed.

2. Transmissivity and duality in pure $Z(N)$ models

Let us consider a site (0-simplex) with which we associate a $Z(N)$ random variable $S \equiv \exp(\mathrm{i}2\pi n/N)$ where $n=0,1,2,\ldots,N-1$. Then we construct a bond (1-simplex) by joining two such sites (denoted 1 and 2) and we associate with it the $Z(N)$ random variable A_1 (the subscript 1 refers to 1-simplex) defined by

$$A_1 \equiv S_1^* S_2 = \exp[\mathrm{i}2\pi(n_2-n_1)/N].$$

Let $p^{(\alpha)}$ be the probability that this variable takes the value $\mathrm{e}^{\mathrm{i}2\pi\alpha/N}$. We define the N-dimensional vector *transmissivity* $\boldsymbol{t}$ through its components given by

$$t^{(\alpha)} = \sum_{\beta=0}^{N-1} p^{(\beta)} \exp\left(\mathrm{i}\frac{2\pi}{N}\alpha\beta\right) \qquad (\alpha=0,1,2,\ldots,N-1); \tag{5}$$

hence

$$p^{(\beta)} = \frac{1}{N}\sum_{\alpha=0}^{N-1} t^{(\alpha)} \exp\left(-\mathrm{i}\frac{2\pi}{N}\alpha\beta\right) \qquad (\beta=0,1,2,\ldots,N-1). \tag{5'}$$

Remark that

$$t^{(0)}=1, \qquad t^{(N-\alpha)}=[t^{(\alpha)}]^*, \tag{6}$$

and that $p^{(N-\beta)}=p^{(\beta)}$ $(\forall\beta)$ implies that $t^{(\alpha)}$ is a real quantity $(\forall\alpha)$; the $\{t^{(\alpha)}\}$ are proportional to the $\{\lambda(\alpha+1)\}$ of Wu and Wang (1976). Remark also that in the case of a Potts bond we have that $p^{(0)}=1/[1+(N-1)\exp(-NJ/k_BT)]$ and, for $\beta\neq 0$,

$$p^{(\beta)} = \exp(-NJ/k_BT)/[1+(N-1)\exp(-NJ/k_BT)]; \tag{7}$$

therefore $t^{(\beta)}$ for $\beta\neq 0$ reduces to expression (1).

Let us now calculate the equivalent transmissivity t_s of a *series* array of two $Z(N)$ bonds whose transmissivities are t_1 and t_2. If we take into account that the equivalent probabilities are given by

$$p_s^{(\alpha)} = \sum_{\beta=0}^{N-1} p_1^{(\beta)} p_2^{(\alpha-\beta)} \tag{8}$$

we immediately obtain that

$$t_s^{(\alpha)} = t_1^{(\alpha)} t_2^{(\alpha)} \qquad (\alpha = 0, 1, 2, \ldots, N-1). \tag{9}$$

If we have instead a *parallel* array, the equivalent probabilities are given by

$$p_p^{(\alpha)} = p_1^{(\alpha)} p_2^{(\alpha)} \Big/ \sum_{\beta=0}^{N-1} p_1^{(\beta)} p_2^{(\beta)} \tag{10}$$

which provides the following relations:

$$t_p^{(\alpha)D} = t_1^{(\alpha)D} t_2^{(\alpha)D} \qquad (\alpha = 0, 1, 2, \ldots, N-1) \tag{11}$$

where

$$t_j^{(\alpha)D} \equiv \frac{\sum_{\beta=0}^{N-1} t_j^{(\beta)} \exp(-\mathrm{i}2\pi\alpha\beta/N)}{\sum_{\beta=0}^{N-1} t_j^{(\beta)}} = \frac{p_j^{(\alpha)}}{p_j^{(0)}} \qquad (j = 1, 2, p). \tag{12}$$

Let us stress that through transformation (12) the series and parallel algorithms (equations (9) and (11)) become *one and the same.*

It is interesting to remark that the real quantity

$$\rho \equiv \frac{1}{\sqrt{N}} \sum_{\alpha=0}^{N-1} t^{(\alpha)} = \sqrt{N} p^{(0)}$$

transforms under duality similarly to a resistance (or a conductance), i.e.

$$\rho^D = 1/\rho.$$

Furthermore the quantity

$$\tau \equiv (\sqrt{N}\rho - 1)/(N-1) \tag{13}$$

transforms under duality like the transmissivity of a Potts model (see equation (4)), i.e.

$$\tau^D = (1-\tau)/[1+(N-1)\tau]. \tag{14}$$

Finally we may define another interesting quantity (used in § 3), namely

$$\sigma \equiv \frac{\ln(\sqrt{N}\rho)}{\ln N} = \frac{\ln[1+(N-1)\tau]}{\ln N}. \tag{15}$$

We immediately verify that under duality σ transforms like a probability, i.e.

$$\sigma^D = 1 - \sigma. \tag{16}$$

This variable generalises the s variable introduced in Levy *et al* (1980) and extended by Tsallis (1981) (see also Tsallis and de Magalhães 1981).

We shall now restate on more general grounds what we have said until now (and by the way clarify the nomenclature introduced in equation (12)). Let us consider a square plaquette (hypercubic s-simplex); its border is constituted by four bonds ($2s$ $(s-1)$-simplex). With the ith bond ($(s-1)$-simplex) we associate a $Z(N)$ random variable $S_i=\exp(\mathrm{i}2\pi n_i/N)$ and with the plaquette (s-simplex) we associate another $Z(N)$ variable denoted by $A_2(A_s)$, defined by $A_2\equiv\Pi'_{\{i\}}S_i\equiv S_1^*S_2^*S_3S_4$ ($A_s\equiv\Pi'_{\{i\}}S_i\equiv S_1^*S_2^*\ldots S_s^*S_{s+1}\ldots S_{2s}$) where the prime stands for *oriented* product (this product runs over *all* the bordering bonds (($s-1$)-simplex) if we are dealing with a non-elementary plaquette (s-simplex)). The plaquette (s-simplex) will be said to be *α-frustrated* when $A_2(A_s)$ equals $\mathrm{e}^{\mathrm{i}2\pi\alpha/N}$ with $\alpha=0,1,2,\ldots,N-1$; it is clear that 0-frustrated corresponds to not frustrated. Let $p^{(\alpha)}$ be the probability that the plaquette (s-simplex) is α-frustrated. Through equation (5) we define the transmissivity t associated with the plaquette (s-simplex).

Two plaquettes (s-simplex) denoted 1 and 2 will be said to be in *series* if they share *one and only one* bordering bond (($s-1$)-simplex); the $Z(N)$ random variable associated with this array is obtained by the product $(A_2)_1(A_2)_2((A_s)_1(A_s)_2)$, therefore equations (8) and (9) still hold in the present general picture. Two plaquettes will be said to be in *parallel* if they share the *whole* border; the probability of this array being α-frustrated is still given by equation (10) which implies equations (11) and (12).

It is well known (Yoneya 1978, Savit 1980) that through duality transformation an s-simplex in a d-dimensional original lattice goes to a ($d-s$)-simplex in the dual lattice. Consequently the transmissivity t of that s-simplex in the original lattice is related to the transmissivity (denoted by t^{D}) of the ($d-s$)-simplex in the dual lattice through equation (12).

Let us now perform an application of the present formalism. We shall consider the general ferromagnetic $Z(N)$ bond system in the square lattice; its Hamiltonian $\mathscr{H}$ (or action) is given (Alcaraz and Köberle 1980, 1981) by

$$\frac{\mathscr{H}}{k_{\mathrm{B}}T}=\sum_{\langle i,j\rangle}h(n_i-n_j) \tag{17}$$

with

$$h(n_i-n_j)=K_1-\sum_{\beta=1}^{\bar{N}}K_\beta[(S_iS_j^*)^\beta+\mathrm{cc}]=K_1-\sum_{\beta=1}^{\bar{N}}2K_\beta\cos\left(\frac{2\pi\beta}{N}(n_i-n_j)\right); \tag{18}$$

the sum of equation (17) runs over all the nearest neighbours and $\bar{N}$ is the integer part of $N/2$ if $N\geqslant2$; in the limit $N\to1$, $\bar{N}$ equals one. The probability that $n_1-n_2=\alpha$ (mod N) is given by

$$p^{(\alpha)}=\mathrm{e}^{-h(\alpha)}\bigg/\sum_{\beta=0}^{N-1}\mathrm{e}^{-h(\beta)} \tag{19}$$

which, through equation (5), leads, for $N\geqslant2$, to

$$t^{(\alpha)}=\sum_{\beta=0}^{N-1}\mathrm{e}^{-h(\beta)}\,\mathrm{e}^{\mathrm{i}2\pi\alpha\beta/N}\bigg/\sum_{\beta=0}^{N-1}\mathrm{e}^{-h(\beta)}. \tag{20}$$

Remark that $t^{(\alpha)}=t^{(N-\alpha)}=(t^{(\alpha)})^*$. If we consider the particular case of the Potts model (for $N>2$, $K_1=K_2=\ldots=K_{\bar{N}-1}=\frac{1}{2}(3+(-1)^N)K_{\bar{N}}$, hence $t^{(1)}=t^{(2)}=\ldots=t^{(N-1)}$) we immediately verify that equation (20) recovers equation (1).

If we now substitute equation (20) into equation (12), we obtain

$$t^{(\alpha)\mathrm{D}}=\mathrm{e}^{-h(\alpha)}/\mathrm{e}^{-h(0)}. \tag{21}$$

If finally we invert equation (20) and replace it in equation (21) we obtain

$$t^{(\alpha)\mathrm{D}}=\sum_{\beta=0}^{N-1}t^{(\beta)}\,\mathrm{e}^{-\mathrm{i}2\pi\alpha\beta/N}\bigg/\sum_{\beta=0}^{N-1}t^{(\beta)} \tag{22}$$

which, through notation changes, corresponds precisely to the *exact* dual transformation (Cardy 1980, Alcaraz and Köberle 1980, 1981). In the particular case of the Potts model we immediately verify that equation (22) recovers equation (4). If we take into account the self-duality of the square lattice and the fact we are considering bonds (whose transformed simplex are still bonds) we have that the general self-dual frontier (which contains all the self-dual points and only them) is given by

$$t = t^{\mathrm{D}}. \tag{23}$$

This equation uniquely determines the location of the critical frontier in the region of the parameter space where it is unique (Cardy 1980, Alcaraz and Köberle 1980, 1981).

For the general four-dimensional $Z(N)$ hypercubic lattice gauge model (whose Hamiltonian—invariant through local gauge $Z(N)$ transformation—is analogous to that of equation (17)) as well as for the general three-dimensional $Z(N)$ cubic lattice gauge model including Higgs fields, it is straightforward to verify that equation (12) corresponds precisely to the *exact* dual transformation (Alcaraz and Köberle 1981).

3. Diluted $Z(N)$ models

We shall now consider a bond-diluted version of the model described by Hamiltonian (17)–(18); in other words its coupling constants will now be random variables whose probability distribution is

$$P_K(K_1, K_2, \ldots, K_{\bar{N}}) = (1-p)\prod_{\beta=1}^{\bar{N}} \delta(K_\beta) + p\prod_{\beta=1}^{\bar{N}} \delta(K_\beta - K_\beta^0) \tag{24}$$

where $\{K_\beta^0\}$ are known constants. This distribution immediately leads to the distribution P_{t} for the transmissivities:

$$P_{\mathrm{t}}(t^{(1)}, t^{(2)}, \ldots, t^{(\bar{N})}) = (1-p)\prod_{\beta=1}^{\bar{N}} \delta(t^{(\beta)}) + p\prod_{\beta=1}^{\bar{N}} \delta(t^{(\beta)} - t_0^{(\beta)}) \tag{25}$$

where the $\{t_0^{(\beta)}\}$ are related to $\{K_\beta^0\}$ through equation (20) with $\{K_\beta^0\}$ playing the role of $\{K_\beta\}$. The probability distribution of the dual variable t^{D} is given by

$$P_{\mathrm{t}}^{\mathrm{D}}(t^{(1)\mathrm{D}}, t^{(2)\mathrm{D}}, \ldots, t^{(\bar{N})\mathrm{D}}) = (1-p)\prod_{\beta=1}^{\bar{N}} \delta(t^{(\beta)\mathrm{D}} - 1) + p\prod_{\beta=1}^{\bar{N}} \delta(t^{(\beta)\mathrm{D}} - t_0^{(\beta)\mathrm{D}}) \tag{26}$$

where $\{t_0^{(\beta)\mathrm{D}}\}$ is related to $\{t_0^{(\beta)}\}$ through equation (22). The probability distributions $P_\tau(\tau)$ and $P_\tau^{\mathrm{D}}(\tau^{\mathrm{D}})$ of the variables τ and τ^{D} respectively defined by equations (13) and (14) are given by

$$P_\tau(\tau) = (1-p)\delta(\tau) + p\delta(\tau - \tau_0) \tag{27}$$

and

$$P_\tau^{\mathrm{D}}(\tau^{\mathrm{D}}) = (1-p)\delta(\tau^{\mathrm{D}} - 1) + p\delta\left(\tau^{\mathrm{D}} - \frac{1-\tau_0}{1+(N-1)\tau_0}\right) \tag{28}$$

where τ_0 is related to $\{t_0^{(\beta)}\}$ through equation (13), substituting $t^{(\alpha)}$ *by* $t_0^{(\alpha)}$.

Following along the lines of Tsallis and Levy (1980), Levy *et al* (1980) and Tsallis (1981), we are led to suggest three slightly different approximations of the CF in the region of the parameter space (whose dimensionality is $\bar{N}+1$) where the transition is unique. Our three present proposals are

$$\langle t^{(1)}\rangle_{P_t} = \langle t^{(1)}\rangle_{P_t^D}, \tag{29}$$

$$\langle \tau\rangle_{P_\tau} = \langle \tau\rangle_{P_\tau^D} \tag{30}$$

and

$$\langle \sigma\rangle_{P_\tau} = \langle \sigma\rangle_{P_\tau^D} \tag{31}$$

which, through use of definitions (13) and (15), respectively lead to

$$pt_0^{(1)} = 1 - p + pt_0^{(1)\,D}, \tag{29'}$$

$$p\tau_0 = 1 - p + p\frac{1-\tau_0}{1+(N-1)\tau_0} \tag{30'}$$

and

$$1 + (N-1)\tau_0 = N^{1/2p}. \tag{31'}$$

We remark that in the particular case $p = 1$ (pure model) all three equations (29′), (30′) and (31′) are contained in the *exact* equation (23) (as a matter of fact it is known that for the pure case these equations provide the same information if $N < 6$ (Cardy 1980, Alcaraz and Köberle 1980); if $N \geq 6$ equation (29′) or equation (30′) or equation (31′) cannot univocally determine the self-dual frontier but only a hypersurface that contains it). In the limit $N \to 1$ all three equations lead to one and the same result, namely

$$p\tau_0 = \tfrac{1}{2} \tag{32}$$

which is known to be *exact* (Southern and Thorpe 1979, Turban 1980, Tsallis 1981); we recall that, in the limit $N \to 1$, $t_0^{(1)} = 1 - t_0^{(1)\,D} = \tau_0$. Furthermore, we verify that all three equations provide $\tau_0 = 1$ for $p = \frac{1}{2}$ (pure bond percolation limit) and that no solution exists for $p < \frac{1}{2}$: this result is commonly believed to be *exact* (Southern and Thorpe 1979, Turban 1980, Tsallis 1981, among others) for the Potts model and we conjecture here that it remains true for the more general model presently discussed. The conjectures (29), (30) and (31) recover, for the Potts model, completely analogous conjectures included in Tsallis and Levy (1980), Levy *et al* (1980) and Tsallis (1981) (it is convenient to recall at this point that the present model extends the Potts one only if $N \geq 4$). In these references it is shown that, for the Potts model, the σ conjecture (equation (31)) is numerically more satisfactory than the others (equations (29) and (30)); it is therefore natural to expect that this is still true in the present generalised picture.

From the very beginning we have considered *isotropic* ferromagnetic models, but no major difficulty exists if crystalline anisotropy is included. In the particular case of the square lattice we can follow along the lines of Tsallis (1981) and propose for the approximate CF the following equation:

$$\langle \sigma\rangle_P + \langle \sigma\rangle_{P'^D} = 1 \tag{33}$$

where $P(P')$ is a general probability distribution for the 'horizontal' ('vertical') coupling constants.

We shall now use equations (29), (30) and (31) to discuss the critical frontier of the $Z(4)$ isotropic bond-dilute model in the square lattice. By associating with each site two

Ising variables μ_i and ν_i (μ_i, $\nu_i = \pm 1$) we can write the $Z(4)$ S_i variable as follows:

$$S_i = \frac{1}{\sqrt{2}}(\mu_i \, e^{-i\pi/4} + \nu_i \, e^{i\pi/4}). \tag{34}$$

Consequently the Hamiltonian (17) can be rewritten

$$\frac{\mathcal{H}}{k_B T} = \sum_{\langle i,j \rangle} [K_1 - K_1(\mu_i\mu_j + \nu_i\nu_j) - 2K_2\mu_i\nu_i\mu_j\nu_j]. \tag{35}$$

The relevant transmissivities of this random model are given (through equation (20)) by

$$t_0^{(1)} = \frac{1 - \exp(-4K_1^0)}{1 + 2\exp[-2(K_1^0 + 2K_2^0)] + \exp(-4K_1^0)}, \tag{36}$$

$$t_0^{(2)} = \frac{1 - 2\exp[-2(K_1^0 + 2K_2^0)] + \exp(-4K_1^0)}{1 + 2\exp[-2(K_1^0 + 2K_2^0)] + \exp(-4K_1^0)}, \tag{36'}$$

and

$$\tau_0 = \tfrac{1}{3}(2t_0^{(1)} + t_0^{(2)}). \tag{37}$$

The t, τ and σ conjectures respectively provide

$$p = \frac{1 + 2t_0^{(1)} + t_0^{(2)}}{t_0^{(1)}(1 + 2t_0^{(1)} + t_0^{(2)}) + 2(t_0^{(1)} + t_0^{(2)})}, \tag{38}$$

$$p = \frac{3(1 + 2t_0^{(1)} + t_0^{(2)})}{(2t_0^{(1)} + t_0^{(2)})(5 + 2t_0^{(1)} + t_0^{(2)})}, \tag{39}$$

$$p = \frac{\ln 2}{\ln(1 + 2t_0^{(1)} + t_0^{(2)})}. \tag{40}$$

All three equations provide qualitatively the same surface (ABCDE in figure 1) in the $(p, t_0^{(1)}, t_0^{(2)})$ space. This surface is expected to be a good approximation of the para-ferromagnetic CF in the region where the transition is unique. Let us now consider some limiting cases. In the plane $t_0^{(1)} = 0$ (i.e. $K_1^0 = 0$ and $t_0^{(2)} = [1 - \exp(-2K_2^0)]/[1 + \exp(-2K_2^0)]$) we have a bond-dilute Ising model CF (the associated Ising variable being $\mu_i\nu_i$) which corresponds to the line I_1G of figure 1. We remark that in this case the Hamiltonian (35) is local gauge invariant; therefore, in accordance with Elitzur's (1975) theorem, $\langle \mu_i \rangle = \langle \nu_i \rangle = 0$ (see also Alcaraz and Köberle 1980, 1981) on both sides of the CF. In the plane $t_0^{(2)} = 1$ (i.e. $K_2^0 \to \infty$ and $t_0^{(1)} = [1 - \exp(-4K_1^0)]/[1 + \exp(-4K_1^0)]$) we have two CF. The first of them (line I_2D of figure 1) corresponds to a bond-dilute Ising model (whose coupling constant equals $2K_1^0$) associated with the variable μ_1 or ν_i. The second CF (straight line GD in figure 1) corresponds to the limit of a thermal problem (whose random variable is $\mu_i\nu_i$) which can be considered as a pure bond percolation one. It is then clear that, if the CF is continuous, the surface ABCDE must bifurcate on some line. It is well known that, on the plane $p = 1$, this bifurcation occurs on the Potts model ($t_0^{(1)} = t_0^{(2)} = \frac{1}{3}$; point B in figure 1); it seems plausible that this is still true on the bond-dilute problem (line BD of figures 1 and 2). As a direct consequence of the preceding considerations only the surface ABDE is concerned by equations (38)–(40). In what concerns the line AED of figure 1 we have not succeeded in formulating a clear interpretation (one plausible equation for that CF is $pt_0^{(1)} = \frac{1}{2}$ for the line AE, the line ED being a straight one). To

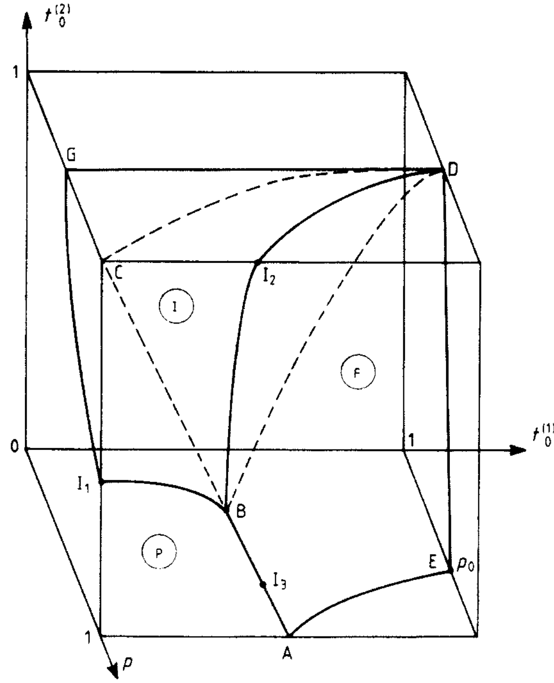

Figure 1. Phase diagram of the bond-dilute $Z(4)$ model in the square lattice (the point E is here located according to the results obtained through the present approximations; it is however possible that the exact p_0 equals $\frac{1}{2}$). B (I_1, I_2 and I_3) is (are) the pure Potts (Ising) critical point(s); the line BD (I_1G and I_2D) corresponds to bond-dilute Potts (Ising) model(s). P, F and I denote the para-, ferromagnetic and intermediate phases.

summarise the preceding analysis, let us say that in the unitary cube of the $(p, t_0^{(1)}, t_0^{(2)})$ space three phases exist, namely the paramagnetic (denoted by P; $Z(4)$ symmetry), the ferromagnetic (denoted by F; completely broken symmetry) and the 'intermediate' (denoted by I; $Z(2)$ symmetry) ones, characterised by:

$$\langle\mu_i\rangle = \langle\nu_i\rangle = \langle\mu_i\nu_i\rangle = 0 \qquad \text{(phase P)},$$

$$\langle\mu_i\rangle \neq 0;\ \langle\nu_i\rangle \neq 0;\ \langle\mu_i\nu_i\rangle \neq 0 \qquad \text{(phase F)},$$

$$\langle\mu_i\rangle = \langle\nu_i\rangle = 0;\ \langle\mu_i\nu_i\rangle \neq 0 \qquad \text{(phase I)}.$$

We can verify directly in the Hamiltonian (35) that $K_2^0 = 0$ (hence $t_0^{(1)} = (t_0^{(2)})^{1/2} = [1-\exp(-2K_1^0)]/[1+\exp(-2K_1^0)]$) corresponds to the bond-dilute Ising model. In this case equations (38) and (40) recover previous results (Nishimori 1979, Tsallis and Levy 1980, Levy *et al* 1980, Tsallis 1981).

In table 1 the most relevant numerical results are presented; we remark that the σ conjecture is globally rather better than the t one which in turn is better than the τ one. In figure 3 we have presented the critical frontiers associated with different ratios $K_2^{(0)}/K_1^{(0)}$; the errors are expected to be not bigger than the graphical widths.

The σ conjecture seems to be (see table 1) asymptotically *exact* in the limit $p \to \frac{1}{2}$ (neighbourhood of point D of figure 1); it provides

$$\left.\frac{d\tau_0}{dp}\right|_{p=1/2} = \frac{1}{3}\left.\frac{d(2t_0^{(1)} + t_0^{(2)})}{dp}\right|_{p=1/2} = -\frac{16}{3}\ln 2 \qquad (41)$$

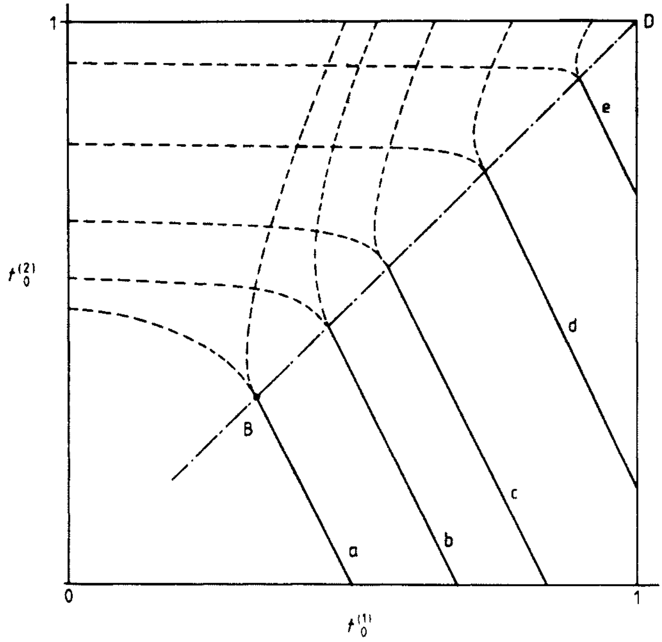

Figure 2. Fixed p sections of the phase diagram of figure 1. (a) $p=1$; (b) $p=0.8$; (c) $p=0.7$; (d) $p=0.6$; (e) $p=0.53$. The line BD corresponds to the bond-dilute Potts model.

which recovers the *exact* answers for the Potts ($K_2^0/K_1^0=\frac{1}{2}$) and Ising ($K_2^0/K_1^0=0$) models. Equation (41) leads to an interesting consequence: for *all* ferromagnetic models satisfying $K_2^0/K_1^0<\frac{1}{2}$, $(\mathrm{d}t_1/\mathrm{d}p)_{p=1/2}$ equals $-4\ln 2$ for fixed ratio K_2^0/K_1^0, whereas for $K_2^0/K_1^0=\frac{1}{2}$, $(\mathrm{d}t_1/\mathrm{d}p)_{p=1/2}$ equals $-\frac{16}{3}\ln 2$.

4. Conclusion

We have introduced, for the general $Z(N)$ s-simplex d-dimensional lattice model, convenient variables (transmissivities) which in series composition behave like probabilities. In what concerns parallel arrays it is possible, through a convenient transformation, to put the parallel composition algorithm in the same form as that of the series case. We have exhibited that this transformation is *precisely* the well known duality transformation.

The simplicity of the present algorithms enables quite plausible approximations to be made for the critical frontiers for random $Z(N)$ models. In order to illustrate this type of conjecture, the square lattice $Z(4)$ bond-dilute ferromagnetic model has been discussed in detail. The phase diagram (see figure 1) exhibits, besides the usual para- and ferromagnetic phases, an intermediate one which is characterised by a partial breakdown of the $Z(4)$ symmetry. A numerically interesting result is the $p=\frac{1}{2}$ limiting slope (equation (41)) which, within the present context, is expected to be exact.

We have seen in § 2 that the functional form of the relevant transformations does not depend on s (order of the s-simplex). Consequently the conjectural picture presented in § 3 should hold for general $Z(N)$ $d/2$-simplex-random ferromagnetic models in d-dimensional hypercubic lattices, thus reinforcing the common belief that the gauge four-dimensional systems are very similar to the bond two-dimensional ones. In particular for the $d/2$-simplex-*dilute* model the ferromagnetic phase disappears at a probability $p=\frac{1}{2}$. This remark suggests the possibility for defining a generalised s-simplex percolation whose critical probability is expected to be $\frac{1}{2}$ for $s=d/2$ and 1 for $s=d$, and which possibly corresponds to a generalisation of the Kasteleyn and Fortuin (1969) $N\to 1$ limit.

Table 1. Relevant quantities (calculated through the t, τ and σ conjectures) associated with the phase diagram represented in figure 1 (where the point E is located at $p=p_0$). See the text for the values followed by (?). (a) Wu and Lin 1974; (b) Sykes and Essam 1963; (c) Baxter 1973; (d) Southern and Thorpe 1979; (e) Kramers and Wannier 1941; (f) Domany 1978; (g) Harris 1974.

Conjectures		t (equation (38))	τ (equation (39))	σ (equation (40))	Exact
p_0		$\frac{3}{5}=0.6$	$\frac{9}{14}\simeq 0.64$	$\frac{\ln 2}{\ln 3}\simeq 0.63$	$\frac{1}{2}$ (?)
Plane $t_0^{(1)}=1$	$-\left.\frac{dt_0^{(2)}}{dp}\right\|_{p=\frac{1}{2},\ t_0^{(2)}=1}$	16	$\frac{48}{5}=9.6$	$16\ln 2\simeq 11.1$	∞ (?)
	$-\left.\frac{dt_0^{(2)}}{dp}\right\|_{p=p_0,\ t_0^{(2)}=0}$	$\frac{25}{4}=6.25$	$\frac{196}{39}\simeq 5.03$	$\frac{3(\ln 3)^2}{\ln 2}\simeq 5.22$	∞ (?)
Plane $t_0^{(2)}=0$	$-\left.\frac{dt_0^{(1)}}{dp}\right\|_{p=1,\ t_0^{(1)}=\frac{1}{2}}$	$\frac{2}{3}\simeq 0.67$	$\frac{3}{4}=0.75$	$\ln 2\simeq 0.69$	$\frac{1}{2}$ (?)
	$-\left.\frac{dt_0^{(1)}}{dp}\right\|_{p=p_0,\ t_0^{(1)}=1}$	$\frac{25}{11}\simeq 2.27$	$\frac{98}{39}\simeq 2.51$	$\frac{3(\ln 3)^2}{2\ln 2}\simeq 2.61$	2 (?)
Plane $p=1$	$-\left.\frac{dt_1}{dt_2}\right\|_{t_0^{(1)}\in[\frac{1}{3},\frac{1}{2}]}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ (a)
Plane $t_0^{(1)}=t_0^{(2)}=t_0$	p_c	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ (b)
	t_c	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$ (c)
	$-\left.\frac{dt_0}{dp}\right\|_{p=\frac{1}{2},\ t_0=1}$	$\frac{16}{5}=3.2$	$\frac{16}{5}=3.2$	$\frac{16}{3}\ln 2\simeq 3.70$	$\frac{16}{3}\ln 2$ (d)
	$-\left.\frac{dt_0}{dp}\right\|_{p=1,\ t_0=\frac{1}{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}\ln 2\simeq 0.46$	$\frac{1}{3\ln 2}\simeq 0.48$ (d)
Surface $t^{(2)}=(t^{(1)})^2\equiv t_0^2$	p_c	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$ (b)
	t_c	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$	$\sqrt{2}-1$ (e)
	$-\left.\frac{dt_0}{dp}\right\|_{p=\frac{1}{2},\ t_0=1}$	$\frac{8}{3}\simeq 2.67$	$\frac{12}{5}=2.4$	$4\ln 2\simeq 2.77$	$4\ln 2$ (f)
	$-\left.\frac{dt_0}{dp}\right\|_{p=1,\ t_0=\sqrt{2}-1}$	$\frac{1}{2}$	$\frac{3}{4\sqrt{2}}\simeq 0.530$	$\frac{\ln 2}{\sqrt{2}}\simeq 0.490$	$(6\sqrt{2}-8)\simeq 0.485$ (g)

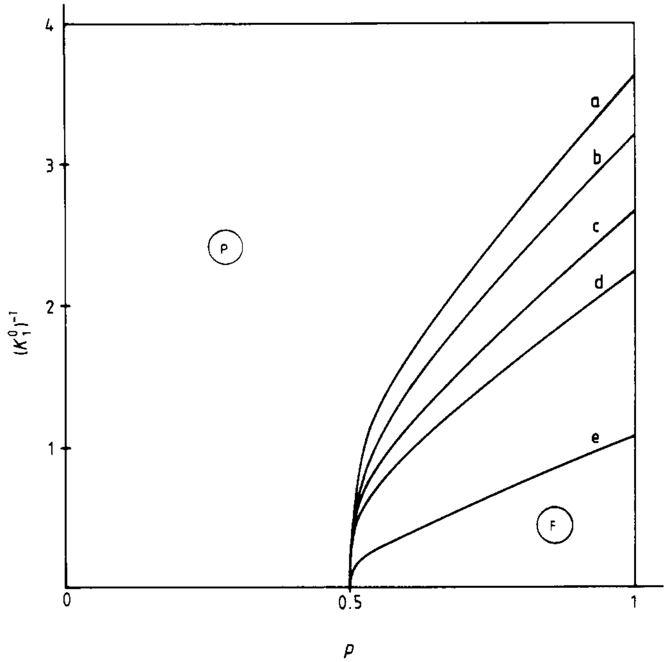

Figure 3. Fixed K_2^0/K_1^0 ratio sections of the phase diagram of the bond-dilute $Z(4)$ model in the square lattice. (a) $K_2^0/K_1^0=0.5$ (Potts); (b) $K_2^0/K_1^0=0.3$; (c) $K_2^0/K_1^0=0.25$; (d) $K_2^0/K_1^0=0$ (Ising); (e) $K_2^0/K_1^0=-0.3$. P and F denote the para- and ferromagnetic phases.

Acknowledgment

One of us (FCA) acknowledges hospitality he received at the Centro Brasileiro de Pesquisas Físicas/CNPq during the visit when this work was done.

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